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Berry's phase, Hannay's angle and quantum normal forms

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Abstract. For any polynomial perturbation of the generalized harmonic oscillator the Berry phase is explicitly computed, in all orders of perturbation theory, as a polynomial in $\hbar$ whose term of order zero is the Hannay angle. This construction is a direct application of the technique of quantum normal forms.

1. Introduction and statement of the results

The reduction of the geometrical phase developed by quantum systems upon adiabatic variation in external parameters along any closed path C, i.e. Berry's phase [1], to the corresponding classical object, i.e. Hannay's angle [2], has been first established by Berry [3] through a quantization formula of the Bohr–Sommerfeld type. More precisely, assuming that the underlying mechanical system is classically integrable, Berry proved the validity of the following *semiclassical* relation between Hannay's angle $\theta(I, C)$ and Berry's phase $\gamma(C)$

$$\theta_k(I, C) = -\frac{\partial\gamma_n(C)}{\partial n_k} = \frac{\partial\gamma_n(C)}{\partial I_k} \qquad k = 1, \ldots, l. \tag{1.1}$$

Here l is the number of degrees of freedom of the system, $I = (I_1, \ldots, I_l)$ the action vector, $n = (n_1, \ldots, n_l)$ the quantum number defining the bound states, related to the actions by Bohr–Sommerfeld quantization $I_j = (n_j + \sigma_j)\hbar$ for $j = 1, \ldots, l$ and where σ_j, the Maslov index, is one and the same half integer for all $j = 1, \ldots, l$.

In the particular example of the generalized harmonic oscillator, namely the system with one degree of freedom described by the quadratic Hamiltonian

$$H_0 = \tfrac{1}{2}[xq^2 + 2yqp + zp^2] \tag{1.2}$$

where the parameters x, y, z have to be considered as slowly varying functions of the time t with $xz - y^2 > 0$, the Bohr–Sommerfeld quantization is exact, and therefore the same is true for the relation (1.1) between Hannay's angle and Berry's phase.

The problem of examining the validity of (1.1) beyond the semiclassical approximation, i.e. of computing the quantum corrections to it or, equivalently, of generating the full semiclassical asymptotics of Berry's phase beyond the term of order 0 in $\hbar$, namely Hannay's angle, has been considered in [4–6] and solved, through microlocal analysis methods, in the one degree of freedom case; in several degrees of freedom an asymptotic expansion up to order $\hbar$ is obtained under the assumption of *quantum integrability*, namely that the Schrödinger operator on $L^2(\mathbb{R}^l)$ can be written as a function of l commuting one-dimensional operators.

The purpose of this paper is to show that, provided all orders in perturbation theory are considered, similar results can also be obtained by a much simpler technique for a large class of non-integrable systems.

More precisely, we consider here the quasi-integrable system defined by any polynomial perturbation (of even degree) of the generalized harmonic oscillator in l degrees of freedom, i.e. the following Hamiltonian family defined on $\mathbb{R}^{2l}$

$$H(P,Q;X,Y,Z;\epsilon) := H_0(P,Q;X,Y,Z) + \epsilon V(P,Q;X,Y,Z) \tag{1.3}$$

where the canonical variables $P = (p_1, \ldots, p_l)$, $Q = (q_1, \ldots, q_l)$ are vectors in $\mathbb{R}^l$ as well as the external parameters $X = (x_1, \ldots, x_l)$, $Y = (y_1, \ldots, y_l)$, $Z = (z_1, \ldots, z_l)$, ϵ is the perturbation strength and

$$\begin{aligned} H_0(P,Q;X,Y,Z) &:= \tfrac{1}{2}\sum_{k=1}^{l}[x_k q_k^2 + 2y_k q_k p_k + z_k p_k^2] \\ V(P,Q;X,Y,Z) &= \sum_{k_1+\cdots+k_l=0}^{2m} V_{k_1,\ldots,k_l}(X,Y,Z) q_1^{k_1}\cdots q_l^{k_l} \end{aligned} \tag{1.4}$$

under the following assumptions:

(1) V takes positive values on the unit sphere in $\mathbb{R}^l$.
(2) There are no rational relations between the 'frequencies' $x_k z_k - y_k^2 \equiv \omega_k$, i.e. the equation in the integer unknowns $(\nu_1, \ldots, \nu_l) \equiv \nu \in \mathbb{Z}^l$

$$\nu_1(x_1 z_1 - y_1^2) + \cdots + \nu_l(x_l z_l - y_l^2) \equiv \langle \nu, \omega\rangle = 0$$

has only the trivial solution $\nu_1 = \cdots = \nu_l = 0$.

In quantum mechanics, the (Weyl) quantization of the classical Hamiltonians (1.3), (1.4) yields a family of Schrödinger operators acting in $L^2(\mathbb{R}^l)$: under the canonical quantization rule $q_k \mapsto \widehat{q}_k$, $p_k \mapsto \widehat{p}_k$ for $k = 1, \ldots, l$, where $\widehat{q}_k$ is the maximal multiplication operator by q_k in $L^2(\mathbb{R}^l)$ and $\widehat{p}_k$ the maximal differentiation operator generated by $-i\hbar\partial_{q_k}$ the Schrödinger operator family $T(\epsilon; X, Y, Z)$ quantizing $H(P,Q;X,Y,Z;\epsilon)$ is defined as follows

$$T(\epsilon; X,Y,Z) = T_0(X,Y,Z) + \epsilon T_1(P,Q;X,Y,Z) \tag{1.5}$$

where T_0 is the maximal operator in $L^2(\mathbb{R}^l)$ generated by

$$T_0(X,Y,Z)u = \tfrac{1}{2}\sum_{k=1}^{l}[x_k\widehat{q}_k^2 + y_k\widehat{q}_k\widehat{p}_k + y_k\widehat{p}_k\widehat{q}_k + z_k\widehat{p}_k^2]u \tag{1.6}$$

and the perturbation T_1 is the maximal multiplication operator generated by the function V. These definitions make T_0 and V self-adjoint in $L^2(\mathbb{R}^l)$, as well as, by assumption (1), T_1, $\{T(\epsilon) : \epsilon > 0\}$ if defined on the maximal domain $D(T_0) \cap D(T_1)$ (see e.g. [7, section XIII.13]). Moreover $T(\epsilon) : \epsilon \geqslant 0$ has a discrete spectrum. Assumption (2) yields the simplicity of the spectrum of T_0 and hence, by the norm resolvent continuity at $\epsilon = 0$ proved in [8], also of $\{T(\epsilon) : \epsilon > 0\}$, which is required to apply the adiabatic theorem of quantum mechanics (see e.g. [9]). Hence the eigenvalues of $T(\epsilon)$ can be labelled by the same quantum numbers $n = (n_1, \ldots, n_l) : n_k \in \mathbb{N}$, $k = 1, \ldots, l$ labelling the eigenvalues $E_0(\hbar, n) = \langle n + \frac{1}{2}, \omega\rangle\hbar$ of T_0.

We also recall (see e.g. [7, section XII.5]) that under the present assumptions the Rayleigh–Schrödinger perturbation theory for both the eigenvalues and the eigenprojections of $T(\epsilon)$ exists to all orders as a formal power series in ϵ, and the same is true for the canonical perturbation expansion for the Hamiltonian $H(\epsilon)$. We can now formalize the main result of the present paper.

Proposition 1. Let $H(\epsilon)$ and $T(\epsilon)$ be as above, and let assumptions (1) and (2) be fulfilled. Then:

(1) The Hannay angle $\theta_k(I, C)$ admits a formal power series expansion to all orders of the initial point $\theta_k^0(I, C)$, the Hannay angle of H_0; namely, for any $s \in \mathbb{N}$, there are $a_s^k(I, C)$ such that

$$\theta_k(I, C) = \theta_k^0(I, C) + \sum_{s=1}^{\infty} a_s^k(I, C)\epsilon^s. \tag{1.7}$$

(2) For any quantum number $n = (n_1, \ldots, n_l)$ the Berry phase $\gamma_n(C)$ admits a formal power series expansion to all orders of the initial point $\gamma_n^0(C)$; namely, for any $s \in \mathbb{N}$, there are $\gamma_s(n)$ such that

$$\gamma_n(C) = \gamma_n^0(C) + \sum_{s=1}^{\infty} \gamma_s(n; C)\epsilon^s. \tag{1.8}$$

(3) For any $s \in \mathbb{N}$ there are entire functions $I \mapsto \Gamma_s^j(I; C)$ for $j = 0, \ldots, 2m(s-1)$ such that, under Bohr–Sommerfeld quantization $I = n\hbar$

$$\gamma_s(n; C) = \sum_{j=0}^{2m(s-1)} \Gamma_s^j(I, C)|_{I=n\hbar} \cdot \hbar^j \tag{1.9}$$

with

$$-\frac{\partial \Gamma_s^0(n\hbar, C)}{\partial n_k} = -\hbar \frac{\partial \Gamma_s^0(I, C)}{\partial I_k} = a_s^k(I, C) \qquad s \in \mathbb{N}, \quad k = 1, \ldots, l. \tag{1.10}$$

Remarks.
(1) (1.1), (1.9) and (1.10) yield the semiclassical asymptotics for Berry's phase to all orders in perturbation theory, namely

$$-\frac{\partial \gamma_s(n)}{\partial n_k} = a_s^k - \sum_{j=1}^{2m(s-1)} \frac{\partial \Gamma_s^j(n)}{\partial n_k} \hbar^j. \tag{1.11}$$

(2) The proof of the proposition is based on a straightforward adaptation to the present situation of the quantum normal form developed in [10, 11] (see also [12, 13]) as the exact quantization of the classical Birkhoff one. This technique essentially allows us to determine, to all orders of perturbation theory, all corrections to the Bohr–Sommerfeld quantization rule and this makes it possible to take over directly the argument of Berry valid for the generalized harmonic oscillator recalled above.
(3) The form (1.3) is the form assumed by any smooth (parameter-dependent) Hamiltonian system in l degrees of freedom near any stable, non-resonant equilibrium point provided the terms of order at least $2m + 1$ in the Taylor expansion are neglected.
(4) The quantization condition for the actions appearing in (3), implied by the quantum normal form, is the original Bohr–Sommerfeld one, $I = n\hbar$, and not the somewhat more familiar $I = (n + \frac{1}{2})\hbar$. The factor $\frac{1}{2}$, the Maslov index, comes from the WKB connection formula which is never used in this context.
(5) The non-resonance condition assumption (2) not only guarantees the simplicity of the spectrum of the Schrödinger operator $T(\epsilon) : \epsilon \geqslant 0$, but is also necessary for the very existence of perturbation theory, both classical and quantum, because its violation would bring in a zero denominator.

In the next section we will prove the 'classical part' of the proposition, namely part (1), and in section 3 the 'quantum part' namely parts (2) and (3) will be proved.

2. Classical normal form. Perturbation expansion of Hannay's angle

As already mentioned, the purpose of this section is to prove part (1) of proposition 1. The main preliminary result in this direction is the construction of a suitable normal form for H_ϵ. To this end, let us first recall the definition of the relevant canonical variables.

(a) For any $k = 1, \ldots, l$ the transformation C [3]

$$\begin{aligned} I_k(p_k.q_k; x_k, y_k, z_k) &= \frac{1}{2\omega_k}[x_k q_k^2 + 2y_k q_k p_k + z_k p_k^2] \\ \theta_k(p_k.q_k; x_k, y_k, z_k) &= \tan^{-1}\left[\frac{z_k}{\omega_k}\left(\frac{p_k}{q_k} + \frac{y_k}{z_k}\right)\right] \end{aligned} \tag{2.1}$$

is a canonical bijection between $\mathbb{R}^2$ and $\mathbb{R} - \{0\} \times \mathbb{T}^1$, whose inverse C^{-1} is

$$\begin{aligned} q_k(I_k, \theta_k, x_k, y_k, z_k) &= \sqrt{\frac{2z_k I_k}{\omega_k}} \cos\theta_k \\ p_k(I_k, \theta_k, x_k, y_k, z_k) &= -\sqrt{\frac{2zI_k}{\omega_k}}\left(\frac{y_k}{z_k}\cos\theta_k + \frac{\omega_k}{z_k}\sin\theta_k\right). \end{aligned} \tag{2.2}$$

The canonical variables $(I, \theta) = (I_1, \ldots, I_l; \theta_1, \ldots, \theta_l)$ are the action-angle variables for H_0, and one has by construction

$$H_0 = \sum_{k=1}^{l} \omega_k I_k. \tag{2.3}$$

(b) The well known complex canonical variables (Bargmann variables) corresponding to (I, θ) are denoted by $(\zeta', \eta') = (\zeta'_1, \ldots, \zeta'_l; \eta'_1, \ldots, \eta'_l)$

$$\zeta'_k = \sqrt{I_k}\mathrm{e}^{\mathrm{i}\theta_k} \qquad \eta'_k = \sqrt{I_k}\mathrm{e}^{-\mathrm{i}\theta_k} \qquad k = 1, \ldots, l. \tag{2.4}$$

They are obviously complex-conjugates; it is however useful to consider them as independent *complex* variables. The expression of H_0 in terms of (ζ', η') is obviously

$$H_0(\zeta', \eta') = \sum_{k=0}^{l} \omega_k \zeta'_k \eta'_k. \tag{2.5}$$

(c) The validity and the canonical character of the following transformations relating the canonical variables (Q, P) to the complex canonical variables (ζ', η'), and the variables (ζ', η') to the complex canonical variables $(\zeta, \eta) := (1/\sqrt{2})(Q + \mathrm{i}P, Q - \mathrm{i}P)$ (using obvious vector notation) is immediately verifiable

$$q_k = \sqrt{\frac{z_k}{\omega_k}}(\zeta'_k + \eta'_k)$$

$$p_k = -\frac{1}{\sqrt{2z_k\omega_k}}[(y_k - i\omega_k)\zeta'_k + (y_k + i\omega_k)\eta'_k] \tag{2.6}$$

$$\begin{aligned}\zeta_k &= \frac{1}{2}\left[\left(\sqrt{\frac{z_k}{\omega_k}} - \sqrt{\frac{\omega_k}{z_k}} - \mathrm{i}\frac{y_k}{\sqrt{z_k\omega_k}}\right)\zeta'_k + \left(\sqrt{\frac{z_k}{\omega_k}} + \sqrt{\frac{\omega_k}{z_k}} - \mathrm{i}\frac{y_k}{\sqrt{z_k\omega_k}}\right)\eta'_k\right]\\ \eta_k &= \frac{1}{2}\left[\left(\sqrt{\frac{z_k}{\omega_k}} + \sqrt{\frac{\omega_k}{z_k}} + \mathrm{i}\frac{y_k}{\sqrt{z_k\omega_k}}\right)\zeta'_k + \left(\sqrt{\frac{z_k}{\omega_k}} - \sqrt{\frac{\omega_k}{z_k}} + \mathrm{i}\frac{y_k}{\sqrt{z_k\omega_k}}\right)\eta'_k\right]\end{aligned} \tag{2.7a}$$

$$\begin{aligned}\zeta'_k &= -\frac{1}{2\mathrm{i}}\sqrt{\frac{z_k\omega_k}{y_k}}\left[\left(1 + \frac{\omega_k}{z_k} - \mathrm{i}\frac{y_k}{z_k}\right)\zeta_k - \left(1 - \frac{\omega_k}{z_k} + \mathrm{i}\frac{y_k}{z_k}\right)\eta_k\right]\\ \eta'_k &= \frac{1}{2\mathrm{i}}\sqrt{\frac{z_k\omega_k}{y_k}}\left[\left(1 + \frac{\omega_k}{z_k} + \mathrm{i}\frac{y_k}{z_k}\right)\zeta_k - \left(1 - \frac{\omega_k}{z_k} - \mathrm{i}\frac{y_k}{z_k}\right)\eta_k\right].\end{aligned} \tag{2.7b}$$

(d) By a standard abuse of notation, we denote by

$$V(\zeta,\eta;X,Y,Z) \qquad V(I,\theta;X,Y,Z) \qquad V(\zeta',\eta';X,Y,Z)$$

the potential V expressed as a function of the canonical variables (ζ,η), (I,θ), (ζ',η'), respectively, by $H_0(\zeta,\eta)$ the Hamiltonian H_0 expressed as a function of (ζ,η), and therefore by $H_\epsilon(I,\theta)$, $H_\epsilon(\zeta',\eta')$, $H_\epsilon(\zeta,\eta)$ the perturbed Hamiltonian when re-expressed in the corresponding variables.

(e) We remark that $H_0(\zeta,\eta)$ can be written in the form

$$H_0(\zeta,\eta) = \sum_{k=1}^{l} \omega_k I_k(\zeta_k,\eta_k) \tag{2.8a}$$

where, by (2.7b)

$$\begin{aligned}I_k(\zeta_k,\eta_k) = \frac{1}{4}\frac{z_k\omega_k}{y_k}&\left[\left(1 + \frac{\omega_k}{z_k} - \mathrm{i}\frac{y_k}{z_k}\right)\zeta_k - \left(1 - \frac{\omega_k}{z_k} + \mathrm{i}\frac{y_k}{z_k}\right)\eta_k\right]\\ &\times\left[\left(1 + \frac{\omega_k}{z_k} + \mathrm{i}\frac{y_k}{z_k}\right)\zeta_k - \left(1 - \frac{\omega_k}{z_k} - \mathrm{i}\frac{y_k}{z_k}\right)\eta_k\right].\end{aligned} \tag{2.8b}$$

We can now state the preliminary result mentioned above.

Lemma 1. For any $N \in \mathbb{N}$ there is a canonical transformation χ_ϵ, depending holomorphically on the canonical variables and on ϵ, mapping $H_\epsilon(\zeta',\eta')$ into the Hamiltonian

$$K_\epsilon(\zeta',\eta') := H_0(\zeta',\eta') + \sum_{j=1}^{N} K_j(\zeta'\eta')\epsilon^j + \mathrm{O}(\epsilon^{N+1}) \tag{2.9a}$$

where, as above, $H_0(\zeta',\eta')$ is defined as in (2.5) and

$$\zeta'\eta' = (\zeta'_1\eta'_1,\ldots,\zeta'_l\eta'_l). \tag{2.9b}$$

Remarks.
(1) (2.9) is the normal form expressed in the canonical coordinates (ζ', η'). By the canonical transformations (2.1), (2.2), (2.4), (2.6) and (2.7) the canonically equivalent normal forms expressed in the canonical coordinates (I, θ) and (ζ, η) are written as

$$K_\epsilon(I,\theta) := H_0(I) + \sum_{j=1}^{N} K_j(I)\epsilon^j + \mathrm{O}\,(\epsilon^{N+1}) \tag{2.10a}$$

$$K_\epsilon(\zeta,\eta) := H_0(\zeta,\eta) + \sum_{j=1}^{N} K_j(I(\zeta,\eta)) + \mathrm{O}\,(\epsilon^{N+1}). \tag{2.10b}$$

In (2.10a), $I = (I_1, \dots, I_l)$ and $H_0(I)$ is defined in (2.3). In (2.10b), $I(\zeta,\eta) = (I_1(\zeta_1,\eta_1), \dots, I_l(\zeta_l,\eta_l))$ where $I_k(\zeta_k,\eta_k)$, $k = 1, \dots, l$ is given by (2.8b).
(2) The dependence on the external parameters X, Y, Z is not explicitly displayed because it is contained in the definition of the canonical variables (I, θ) and hence (ζ', η'), (ζ, η) (see (2.3)–(2.8b)).
(3) As we will see later on, the form (2.10a) yields the expansion for Hannay's angle by a direct application of Berry's argument, while the form (2.10b) admits the exact quantization yielding the quantum normal form.

Proof. The argument is, as in [11], a straightforward application of canonical perturbation theory generated by the Lie algorithm written in the Bargmann variables, and could be omitted; we include it here for convenience of exposition because the perturbative computation of Berry's phase is based on the exact quantization of this procedure.

The canonical perturbation theory is generated by looking for the canonical transformation $(\zeta_1, \eta_1) = \chi_\epsilon(\zeta', \eta')$ such that $H_\epsilon \circ \chi_\epsilon = K(\zeta'\eta', \epsilon)$ where $\zeta'\eta'$ is given by (2.9b). (Note that we keep the notation (ζ', η') to denote the 'new' canonical variables.) The Lie algorithm (see e.g. [14, 15]) consists in determining χ_ϵ as the flow of an auxiliary, non-autonomous Hamiltonian $w_\epsilon(\zeta', \eta')$ in which ϵ plays the role of 'time'. Given any holomorphic observable $f(\zeta', \eta')$ on $\mathbb{C}^{2l}$, and any holomorphic Hamiltonian $w(\zeta', \eta')$, denote by t_ϵ the composition operator with χ_ϵ : $(t_\epsilon f)(\zeta', \eta') = (f \circ \chi_\epsilon)(\zeta', \eta')$ and by $(\mathcal{L}_w f)(\zeta', \eta') = \{f, w\}(\zeta', \eta')$ the Lie derivative along the flow generated by w. The equation to be solved is therefore

$$t_\epsilon H_\epsilon = K(\zeta'\eta', \epsilon). \tag{2.11}$$

We remark now that, for any holomorphic observable f we can write

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\zeta', \eta') = t_\epsilon(\{f, w_\epsilon\})(\zeta', \eta') = t_\epsilon(\mathcal{L}_w f)(\zeta', \eta')$$

because ϵ is the 'time' and χ_ϵ the flow generated by the Hamiltonian w_ϵ

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\eta' = \mathrm{i}\nabla_{\zeta'} w_\epsilon \qquad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\zeta' = -\mathrm{i}\nabla_{\eta'} w_\epsilon \tag{2.12}$$

which takes the initial datum (ζ_1, η_1) at (ζ', η') at 'time' ϵ. Expand now w_ϵ, t_ϵ, $K(\zeta'\eta', \epsilon)$ in a (formal) power series of ϵ

$$w_\epsilon = \sum_{\ell=0}^{\infty} \epsilon^\ell w_{\ell+1} \qquad t_\epsilon = \sum_{\ell=0}^{\infty} \epsilon^\ell t_\ell \qquad K(\zeta'\eta', \epsilon) = \sum_{\ell=0}^{\infty} \epsilon^\ell K_\ell(\zeta'\eta') \tag{2.13}$$

whence $\mathrm{d}t_\epsilon/\mathrm{d}\epsilon = \sum_{\ell=0}^{\infty}\epsilon^{\ell-1}\ell t_\ell$. Inserting (2.13) in this last equation and equating the coefficients of the resulting powers of ϵ we get

$$\ell t_\ell = \sum_{j=1}^{\ell} t_{\ell-j}\mathcal{L}_{w_j} \qquad t_0 = I_d \tag{2.14}$$

so that, expanding both sides of (2.11)

$$t_\ell K_0 + t_{\ell-1}V = K_\ell \qquad \ell = 1, 2, \ldots, \quad K_0 = H_0(\zeta', \eta') \tag{2.15}$$

whence, by (2.14)

$$\frac{1}{\ell}\{H_0, w_\ell\} + V_\ell = K_\ell \qquad \ell = 1, 2, \ldots \tag{2.16}$$

$$V_1 = V \qquad V_\ell = t_{\ell-1}V + \frac{1}{\ell}\sum_{j=1}^{\ell-1} t_{\ell-j}\mathcal{L}_{w_j}H_0 \qquad \ell = 2, 3, \ldots. \tag{2.17}$$

To solve these recurrent equations we look for the Taylor expansion (in the (ζ', η') variables) of the unknown functions: given the Taylor expansion of V_ℓ

$$V_\ell = \sum_{\alpha,\beta=0}^{\infty} V_{\alpha,\beta}^{(\ell)}\zeta'^\alpha\eta'^\beta \qquad V_1 \equiv V = \sum_{\alpha,\beta=0}^{2m} V_{\alpha,\beta}\zeta'^\alpha\eta'^\beta \tag{2.18a}$$

whose construction only involves w_j and K_j up to $j = \ell - 1$, the coefficients of the Taylor expansions of w_ℓ and K_ℓ

$$w_\ell = \sum_{\alpha,\beta=0}^{\infty} w_{\alpha,\beta}^{(\ell)}\zeta'^\alpha\eta'^\beta \qquad K_\ell = \sum_{\alpha=0}^{\infty} K_\alpha^{(\ell)}\zeta'^\alpha\eta'^\alpha \tag{2.18b}$$

are given by

$$w_{\alpha,\beta}^{(\ell)} = \mathrm{i}\frac{V_{\alpha,\beta}^{(\ell)}}{\langle\omega, (\beta-\alpha)\rangle} \qquad \alpha \neq \beta \qquad K_\alpha^{(\ell)} = V_{\alpha,\alpha}^{(\ell)} \tag{2.18c}$$

where, as usual α, β are multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_l)$, $\beta = (\beta_1, \ldots, \beta_l)$

$$\zeta'^\alpha\eta'^\beta = \zeta_1'^{\alpha_1}\cdots\zeta_l'^{\alpha_l}\eta_1'^{\beta_1}\cdots\eta_l'^{\beta_l} \qquad \zeta'^\alpha\eta'^\alpha = (\zeta_1'\eta_1')^{\alpha_1}\cdots(\zeta_l'\eta_l')^{\alpha_l}$$

$\omega = (\omega_1, \ldots, \omega_l)$ and $\langle\omega, (\beta-\alpha)\rangle$ denotes the scalar product between the l-vectors ω and $\alpha - \beta$. Since V is a polynomial, the Taylor series (2.18a, b) actually reduce to finite sums, and the denominators never vanish by the non-resonance condition. □

Proof of proposition 1, part (1). To apply Berry's argument [3, equation (17)] without the slightest change we have only to re-express the original canonical coordinates (Q, P) in terms of the canonical coordinates (I, θ) in which H_ϵ assumes the normal form (2.10a). We have immediately

$$(Q, P) = C^{-1}\circ\chi_\epsilon\left(\sqrt{I}\mathrm{e}^{\mathrm{i}\theta}, \sqrt{I}\mathrm{e}^{-\mathrm{i}\theta}\right) \tag{2.19}$$

where C^{-1} is given by (2.2) and the vector notation $\sqrt{I}e^{i\theta} = (\sqrt{I_1}e^{i\theta_1}, \ldots, \sqrt{I_l}e^{i\theta_l})$ has been used once more. Now

$$\chi_\epsilon(\sqrt{I}e^{i\theta}, \sqrt{I}e^{-i\theta}) = \sum_{\ell=1}^{N} \mathcal{J}\nabla_{(\zeta',\eta')} w_\ell(\sqrt{I}e^{i\theta}, \sqrt{I}e^{-i\theta}) + O(\epsilon^{N+1}) \tag{2.20}$$

where $\mathcal{J}\nabla$ denotes the symplectic gradient. Insertion of (2.19), (2.20) into (18) of [3] yields the result, (with the identification dA = surface element in parameter space)

$$a_s^k(I, C) = -\frac{\partial}{\partial I_k} \iint_{\partial A = C} W_s^k(I; X, Y, Z)\, dA \tag{2.21}$$

and

$$W_s(I; X, Y, Z) = \frac{1}{(2\pi)^l} \oint \nabla_{X,Y,Z} p_k^s(I, \theta; X, Y, Z) \wedge \nabla_{X,Y,Z} q_k^s(I, \theta; X, Y, Z)\, d\theta_k \tag{2.22}$$

where

$$\begin{aligned} p_k^s(I, \theta; X, Y, Z) &= p_k\left(-\frac{\partial}{\partial \zeta_k'} w_s(\sqrt{I_k}e^{i\theta_k}, \sqrt{I_k}e^{-i\theta_k}); X, Y, Z\right) \\ q_k^s(I, \theta; X, Y, Z) &= q_k\left(\frac{\partial}{\partial \zeta_k'} w_s(\sqrt{I_k}e^{i\theta_k}, \sqrt{I_k}e^{-i\theta_k}); X, Y, Z\right). \end{aligned} \tag{2.23}$$

□

3. Quantum normal form. Asymptotics of Berry's phase

We now proceed to the construction of a suitable quantum normal form for the operator family $T(\epsilon)$ along the lines of [11]. We first remark that, since Berry's phase is left invariant by any parameter-independent unitary transformation in the Hilbert space (see e.g. [3, equation (23)]), we are free to choose the representation of the canonical commutation rule. Therefore we will employ the Bargmann representation [16]: this is precisely the representation in which the quantum normal form can be constructed as the exact quantization of the classical one, written in the (ζ, η) variables (equation (2.9b) above). Indeed consider for any $\psi \in L^2(\mathbb{R}^l)$ the integral transform

$$(U\psi)(z) \equiv f(z) = (\sqrt{\pi}\hbar)^{-l/2} \int_{\mathbb{R}^{2l}} \exp\left[-\frac{(z^2 + Q^2)}{2\hbar} + 2\frac{\sqrt{2}\langle z, Q\rangle}{\hbar}\right] \psi(Q)\, dQ \tag{3.1}$$

where $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$, $z^2 = z_1^2 + \cdots + z_l^2$, $Q^2 = Q_1^2 + \cdots + q_l^2$. U is a unitary map between $L^2(\mathbb{R}^l)$ and the Bargmann space $\mathcal{F}_l$ of all entire holomorphic functions u on $\mathbb{C}^l$ such that

$$\|u\|_{\mathcal{F}_l} \equiv \int_{\mathbb{R}^{2l}} |u(z)|^2 \exp\left[-\frac{|z|^2}{\hbar}\right] \prod_{k=1}^{l} dz_k d\overline{z}_k < +\infty. \tag{3.2}$$

It is well known that $\mathcal{F}_l$ yields a representation of the canonical commutation rules in which the standard creation and annihilation operators of the harmonic oscillator

$$a_k = \frac{1}{\sqrt{2}}(\widehat{q}_k - \mathrm{i}\widehat{p}_k) \qquad a_k^\dagger = \frac{1}{\sqrt{2}}(\widehat{q}_k + \mathrm{i}\widehat{p}_k)$$

which are nothing else than the quantization of the classical observables (ζ, η), act as multiplication and differentiation operators, respectively

$$(\widehat{\zeta}_k u)(z) = z_k u(z) \qquad (\widehat{\eta}_k u)(z) = \hbar \frac{\partial u}{\partial z_k}(z). \tag{3.3}$$

Equation (3.3), together with the Weyl quantization rule, yields the canonical quantization of any real-holomorphic classical observable $f(\zeta, \eta)$ as a (formally) self-adjoint operator $\widehat{f}$ in $\mathcal{F}_l$, which can be constructed in the following way (see e.g. [17, section 5.1–2]; without loss of generality we can assume here $l = 1$)

(1) Denote by $\Pi_S(A^m B^n)$ the symmetric product of the mth and nth powers of two non-commuting operators A and B, formally defined as the coefficient of $((m+n)!/m!n!)X^m Y^n$ in the expansion of $(AX + BY)^{m+n}$, $(X, Y) \in \mathbb{R}$.
(2) Denote by $\widehat{u} = \Pi_S(\widehat{\zeta}^m \widehat{\eta}^n)$ the symmetric product quantization (as a maximal operator in $\mathcal{F}_l$) of the classical observable $u(\zeta, \eta) = \zeta^m \eta^n$.
(3) Then if

$$f(\zeta, \eta) = \sum_{m,n=0}^{\infty} f_{mn} \zeta^m \eta^n$$

its canonical (Weyl) quantization $\widehat{f}$ is defined as

$$\widehat{f} = \sum_{m,n=0}^{\infty} f_{mn} \Pi_S(\widehat{\zeta}^m \widehat{\eta}^n). \tag{3.4}$$

Therefore, denoting by $S_0(\hbar)$, $S_1(\hbar)$ and $S(\hbar, \epsilon)$, respectively, the unitary images in $\mathcal{F}_l$ under U of the operators T_0, T_1 and $T(\hbar, \epsilon)$, we have

$$S_0 = \widehat{H_0(\zeta, \eta)} \qquad S_1 = \widehat{V} \qquad S(\epsilon) = \widehat{H_0(\zeta, \eta)} + \widehat{V}. \tag{3.5a}$$

We further remark that

$$\begin{aligned} S_0(X, Y, Z) = {} & S_0^1(x_1, y_1, z_1) \otimes I \otimes \cdots \otimes I + I \otimes S_0^2(x_2, y_2, z_2) \otimes \cdots I + \cdots \\ & + I \otimes \cdots \otimes S_0^l(x_l, y_l, z_l) \end{aligned} \tag{3.5b}$$

where, for $k = 1, \ldots, l$

$$S_0^k(\hbar; x_k, y_k, z_k) = \omega_k \widehat{I_k(\zeta, \eta)} \tag{3.5c}$$

and that these operators act as the maximal differential operators in $\mathcal{F}_l$ generated by the differential expressions

$$S_0^k(\hbar; x_k, y_k, z_k) f(z_k) = \hbar(\omega_k z_k \partial_{z_k}) f(z_k) \tag{3.6a}$$

$$\begin{aligned} & (\widehat{V}(X, Y, Z)u)(z) \\ & \quad = \sum_{k_1 + \cdots + k_l = 0}^{2m} V_{k_1, \ldots, k_l}(X, Y, Z) \Pi_S[(z_1 + \hbar\partial_{z_1})^{k_1} \cdots (z_l + \hbar\partial_{z_l})^{k_l}] u(z) \end{aligned} \tag{3.6b}$$

where, as above (equation (1.4))

$$V(\xi,\eta;X,Y,Z)=\sum_{k_1+\cdots+k_l\leqslant 2m} V_{k_1,\ldots,k_l}(X,Y,Z)(\xi_1+\eta_1)^{k_1}\cdots(\xi_l+\eta_l)^{k_l}. \tag{3.6c}$$

We now go on, following [11] to which the reader is referred for the complete details, to the construction of the quantum normal form, i.e. of the Rayleigh–Schrödinger perturbation theory through the 'exact' quantization of the classical Lie algorithm recalled above. The first step is represented by the construction of the Rayleigh–Schrödinger perturbation theory by a procedure formally analogous to canonical perturbation theory. Once more we omit to indicate the explicit dependence on the external parameters (X, Y, Z) because it is not essential in the following argument.

In Rayleigh–Schrödinger perturbation theory we want to diagonalize $S(\hbar,\epsilon)$ on the unperturbed basis by looking for a unitary operator X_ϵ such that $X_\epsilon S(\hbar,\epsilon)X_\epsilon^{-1} = D_\epsilon$, where D_ϵ commutes with S_0. Denoting by $\tilde{W}_\epsilon = \int_0^\epsilon W_x\,\mathrm{d}x$ the self-adjoint generator of X_ϵ, $X_\epsilon = \exp(-(\mathrm{i}/\hbar)\tilde{W}_\epsilon)$ we formally have, for any operator A in $\mathcal{F}_l$, the well known equations

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}X_\epsilon = -\frac{\mathrm{i}}{\hbar}X_\epsilon W_\epsilon \qquad X_0 = I \tag{3.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}X_\epsilon A X_\epsilon^{-1} = \frac{\mathrm{i}}{\hbar}X_\epsilon[A, W_\epsilon]X_\epsilon^{-1}. \tag{3.7b}$$

To generate the operator Rayleigh–Schrödinger perturbation theory we set, in the sense of the formal power series

$$W_\epsilon = \sum_{j=0}^{\infty}\epsilon^j W_{j+1} \qquad X_\epsilon A X_\epsilon^{-1} = \sum_{j=0}^{\infty}\epsilon^j \mathcal{T}_j A \tag{3.8}$$

and

$$L_{W_j}A = [A, W_j] \qquad W_0 = I \qquad L_{W_0} = 0. \tag{3.9a}$$

Inserting (3.9a) and (3.8) in (3.7b) and equating the coefficients of ϵ^s on both sides we get

$$\mathcal{T}_s = \frac{\mathrm{i}}{s\hbar}\sum_{j=1}^{s}\mathcal{T}_{s-j}L_{W_j} \qquad s = 1, 2, \ldots, \qquad \mathcal{T}_0 = Id. \tag{3.9b}$$

Now set $A = S_0(\hbar) + \epsilon V$ and look for the (formal) power series expansion for D_ϵ

$$D_\epsilon = \sum_{\ell=0}^{\infty}\epsilon^\ell D_\ell. \tag{3.10}$$

Then, once more equating the coefficients of the same powers of ϵ the requirement $X_\epsilon S(\hbar,\epsilon)X_\epsilon^{-1} = D_\epsilon$ easily yields, on account of (3.7b), (3.8) and (3.9a,b), the recurrent equations

$$D_0 = S_0(\hbar) \qquad \mathcal{T}_\ell S_0 + \mathcal{T}_{\ell-1}S_1 = D_\ell \qquad \ell = 1, 2, \ldots \tag{3.11a}$$

namely

$$\frac{i}{\ell\hbar}[S_0, W_\ell] + S_\ell = D_\ell \qquad \ell = 1, 2, \dots \tag{3.11b}$$

where

$$V_1 = \widehat{V} = S_1 \qquad V_\ell = \mathcal{T}_{\ell-1}S_1 + \frac{i}{\ell\hbar}\sum_{j=1}^{\ell-1}\mathcal{T}_{\ell-j}L_{W_j}S_0 \qquad \ell = 2, \dots. \tag{3.12}$$

These equations are in turn easily solved on the orthonormal basis

$$\{f_n(z) : n = (n_1, \dots, n_l); n_k = 0, 1, \dots, k = 1, \dots, l\}$$

of the eigenvectors of S_0

$$S_0 f_n = E_0(n, \hbar, \omega) f_n \qquad E_0(n, \hbar, \omega) = \hbar[\langle n, \omega\rangle + \tfrac{1}{2}|\omega|].$$

We have indeed

$$\begin{aligned} &\langle f_m, W_\ell f_n\rangle = i\hbar\ell\frac{\langle f_m, S_\ell f_n\rangle}{E_0(m) - E_0(n)} \qquad m \neq n \\ &\langle f_m, [S_0, W_\ell] f_m\rangle = 0 \\ &\langle f_m, D_\ell f_n\rangle = \langle f_m, S_\ell f_n\rangle\delta_{m,n} \end{aligned} \tag{3.13}$$

because $E_0(m) = E_0(n)$ iff $m = n$ by the simplicity of $\sigma(S_0)$ and the sums contain only a finite number of terms since $S_1 = \widehat{V}$ is a polynomial in the creation and destruction operators $\widehat{q}_k$ and $\widehat{p}_k$. Therefore we can conclude that the (formal) Rayleigh–Schrödinger perturbation theory for the eigenvalues $E_n(\hbar, \epsilon)$ of $S(\hbar, \epsilon)$, and hence of $T(\hbar, \epsilon)$, written as

$$E_n(\hbar, \epsilon) = E_0 + \sum_{\ell=1}^{\infty} E_\ell(n, \hbar)\epsilon^\ell \tag{3.14}$$

exists to all orders, with

$$E_\ell(n, \hbar) = \langle f_n, D_\ell f_n\rangle = \langle f_n, S_\ell f_n\rangle. \tag{3.15}$$

We are now in position to verify part (2) of proposition 1.

Proof of proposition 1, part (2). Let us consider the perturbed eigenfunctions and examine their explicit dependence on the external parameters (X, Y, Z). By construction, for any $N \in \mathbb{N}$ we have

$$\begin{aligned} &S(\hbar, \epsilon; X, Y, Z) f_n^N(z, \hbar, \epsilon; X, Y, Z) \\ &\qquad = E_n^N(\hbar, \epsilon; X, Y, Z) f_n^N(z, \hbar, \epsilon; X, Y, Z) + O(\epsilon^{N+1}) \end{aligned} \tag{3.16a}$$

where

$$f_n^N(z, \hbar, \epsilon; X, Y, Z) = \exp\left[-\frac{i}{\hbar}\sum_{\ell=1}^{N}\frac{W_\ell(X, Y, Z)}{\ell}\epsilon^l\right] f_n(z; X, Y, Z). \tag{3.16b}$$

In fact, since $W_\ell(X,Y,Z)$ is once more a polynomial in the creation and destruction operators, the exponential is well defined on the unperturbed eigenvectors $f_n(z;X,Y,Z)$ by Taylor expansion and this gives (3.16*a*). Now the dependence of V, $E_0(\hbar,n)$ and $f_n(z;X,Y,Z)$ on (X,Y,Z) is obviously holomorphic; therefore the same is true for the dependence of $\langle f_n, S_1 f_m\rangle$, $\langle f_n, W_1 f_m\rangle$, and, by recurrence, for the dependence of $\langle f_n, W_\ell f_m\rangle$ for $\ell = 2,\ldots$. Hence the approximate eigenfunctions $f_n^N(z,\hbar,\epsilon;X,Y,Z)$ admit Taylor expansions in powers of ϵ with coefficients depending holomorphically on (X,Y,Z). Now, by [3, equation (23)], the unitarity of the Bargmann transform and (3.16*a*,*b*), the phase 2-form admits the following representation

$$\mathcal{V}(n;\epsilon;X,Y,Z) = \operatorname{Im}\nabla_{X,Y,Z}\wedge\int_{\mathbb{R}^{2l}}\overline{f}_n^N(z,\hbar,\epsilon;X,Y,Z)\nabla_{X,Y,Z}f_n^N(z,\hbar,\epsilon;X,Y,Z)\,\mathrm{d}z\,\mathrm{d}\overline{z} \tag{3.16c}$$

and this proves the assertion upon expansion in powers of ϵ. □

Remarks.

(1) We note for further reference the following expression for the coefficients $\gamma_s(n,C)$ of the Rayleigh–Schrödinger expansion of Berry's phase in terms of $\mathcal{V}(n;\epsilon;X,Y,Z)$ which follows from (3.16*c*) and [3, equation (20)]:

$$\gamma_s(n,C) = -\iint_{\partial A=C}\mathcal{V}_s(n;X,Y,Z)\cdot\mathrm{d}A \tag{3.16d}$$

where

$$\mathcal{V}_s(n;X,Y,Z) = \operatorname{Im}\nabla_{X,Y,Z}\wedge\frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}\epsilon^s}\Big|_{\epsilon=0} \times\int\overline{f}_n^N(z,\hbar,\epsilon;X,Y,Z)\nabla_{X,Y,Z}f_n^N(z,\hbar,\epsilon;X,Y,Z)\prod_{k=1}^{l}\mathrm{d}z_k\,\mathrm{d}\overline{z}_k. \tag{3.16e}$$

(2) The eigenvectors $f_n(z)$ of S_0 are related to the standard basis vectors of $\mathcal{F}_l$, defined as

$$e_n(z) = \frac{{z_1}^{n_1}}{\sqrt{\pi\hbar n_1!}}\cdots\frac{{z_l}^{n_l}}{\sqrt{\pi\hbar n_l!}}\qquad n=(n_1,\ldots,n_l)\qquad n_k=0,1,\ldots \tag{3.16f}$$

by a unitary transformation $\mathcal{U}(X,Y,Z)$: $f_n = \mathcal{U}(X,Y,Z)e_n$. $\mathcal{U}(X,Y,Z)$ is the quantization of the linear canonical transformation (2.8), and depends holomorphically on (X,Y,Z) (for details see [16, section 3f]).

To prove part (3) of proposition 1 we show that the quantum perturbation theory can be realized as the *exact* quantization of the classical one. By this we mean the following: the quantum recurrent equations (3.11*a*,*b*), (3.12) can be formally obtained from the classical ones (2.16) upon replacement of the Poisson brackets by the quantum commutators and of the classical observables by their Weyl quantization. The replacement is formal because the commutator of two quantized observables does not coincide with the quantization of their Poisson brackets. The exact quantization consists precisely in the recursive determination of the needed corrections.

The key preliminary result in this direction is however the verification that, for the particular case of the free Hamiltonian, the operations of quantizing and of taking the Poisson bracket actually *do* commute.

Lemma 2. Let $f(\zeta,\eta)$ be any holomorphic observable defined on $\mathbb{C}^{2l}$, $\widehat{f}$ its (Weyl) quantization, and $S_0(X,Y,Z)$ the quantization of the free Hamiltonian $H_0(\zeta,\eta;X,Y,Z)$ defined in (3.8). Then

$$[\widehat{f}, S_0(X,Y,Z)] = -\mathrm{i}\hbar\widehat{\{f, H_0(\cdot, X, Y, Z)\}}. \tag{3.17}$$

Proof. We apply the explicit equation [11, equations (3.5), (3.6)] for the Moyal bracket of two (Weyl) quantized operators $\widehat{f}, \widehat{g}$ in $\mathcal{F}_l$ corresponding to the holomorphic observables $f(\zeta,\eta), g(\zeta,\eta)$

$$\frac{\mathrm{i}}{\hbar}[\widehat{f},\widehat{g}] = \sum_{j=0}^{\infty}\widehat{\{f,g\}_{(j)}}\hbar^j \tag{3.18a}$$

where $\{f,g\}_{(j)} = 0$, $j = 1, 3, 5\ldots$ and

$$\{f,g\}_{(j)} = 2^{-(j-1)}\mathrm{i}\sum_{s=0}^{j+1}(-1)^s \sum_{|\alpha|=j+1-s, |\beta|=s} \frac{(\partial_\eta^\beta\partial_\zeta^\alpha)f(\partial_\zeta^\beta\partial_\eta^\alpha)g}{\alpha!\beta!} \qquad j = 0, 2, \ldots. \tag{3.18b}$$

Here $|\alpha| = \alpha_1 + \cdots + \alpha_l$, $|\beta| = \beta_1 + \cdots + \beta_l$. Setting $g(\zeta,\eta) = H_0(\zeta,\eta;X,Y,Z)$, by (2.8*a*, *b*) it is immediately verified that $\{f, H_0(\cdot, X, Y, Z)\}_{(j)} = 0\ \forall j > 0$. □

We can now easily adapt the recursive arguments of [11, lemmas 4, 5, 6] to establish the existence of a suitable quantum normal form, namely:

Lemma 3. For any $\ell = 2, \ldots, j = 1, \ldots, 2m(\ell-1)$ there are recursively defined polynomials $(\zeta,\eta) \mapsto V_j^\ell(\zeta,\eta;X,Y,Z)$, $(\zeta,\eta) \mapsto \kappa_j^\ell((I_1(\zeta_1,\eta_1),\ldots,I_l(\zeta_l,\eta_l));X,Y,Z)$, $(\zeta,\eta) \mapsto w_j^\ell(\zeta,\eta;X,Y,Z)$ of degree not exceeding $2m(\ell-1)$ such that

$$S_\ell = \widehat{V_\ell}(\zeta,\eta : X,Y,Z) + \sum_{j=1}^{2m(\ell-1)}\widehat{V_j^\ell}(\zeta,\eta;X,Y,Z)\hbar^j \tag{3.19a}$$

$$W_\ell = \widehat{w_\ell}(\zeta,\eta;X,Y,Z) + \sum_{j=1}^{2m(\ell-1)}\widehat{w_j^\ell}(\zeta,\eta;X,Y,Z)\hbar^j \tag{3.19b}$$

$$D_\ell = \widehat{K_\ell}((I_1(\zeta_1,\eta_1),\ldots,I_l(\zeta_l,\eta_l));X,Y,Z) + \sum_{j=1}^{2m(\ell-1)}\widehat{\kappa_j^\ell}((I_1(\zeta_1,\eta_1),\ldots,I_l(\zeta_l,\eta_l));X,Y,Z)\hbar^j. \tag{3.19c}$$

Proof. Lemma 2 shows that the first quantum homological equation (3.11*b*) is the exact quantization of the corresponding classical equation (2.16), i.e. (once again we omit to indicate the explicit dependence on the external parameters)

$$\frac{\mathrm{i}}{\hbar}[S_0, W_1] + S_1 = D_1 \Longleftrightarrow \widehat{\{H_0, w_1\}} + \widehat{V} = \widehat{K}_1. \tag{3.20}$$

Therefore we can repeat the arguments of [11, lemmas 4, 5, 6] up to the following variants:

(1) The classical homological equation $\{H_0, w_1\} + V = K_1$ has to be solved in the (ζ', η') canonical coordinates, which yield a constant coefficient, first-order partial differential equation as we have seen in the proof of lemma 1. The solution is then re-expressed in the (ζ, η) variables through (2.8*a*, *b*).

(2) The same procedure shows that K_1, and hence by recurrence K_ℓ and also $\kappa_j^\ell : \ell > 1$, $j = 1, \ldots, 2m(\ell - 1)$ depend on (ζ, η) only through $I_1(\zeta_1, \eta_1), \ldots, I_l(\zeta_l, \eta_l)$.

□

A consequence of this normal form is the existence of the semiclassical expansion for any term $\gamma_\ell(n; C)$ of the Rayleigh–Schrödinger expansion of Berry's phase.

Lemma 4. For any $\ell \in \mathbb{N}$, there exist coefficients $\gamma_\ell^j(n; C) : j = 0, \ldots, 2m(\ell - 1)$ such that the following expansion holds

$$\hbar\gamma_\ell(n; C) = \sum_{j=0}^{2m(\ell-1)} \gamma_\ell^j(n; C)\hbar^j. \tag{3.21}$$

Proof. Consider the representation (3.16*d*, *e*) of $\gamma_\ell(n; C)$. By (3.16*b*) we have

$$\mathrm{i}\hbar\nabla_{X,Y,Z} f_n^N(z, \hbar, \epsilon; X, Y, Z) = \left(\nabla_{X,Y,Z} \sum_{p=1}^{N} \frac{W_p(X, Y, Z)}{p}\epsilon^p\right) f_n^N(z, \hbar, \epsilon; X, Y, Z).$$

Now apply the Baker–Campbell–Hausdorff equation to

$$\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{p=1}^{N} \frac{W_p(X, Y, Z)}{p}\epsilon^p\right) \cdot \left(\nabla_{X,Y,Z} \sum_{p=1}^{N} \frac{W_p(X, Y, Z)}{p}\epsilon^p\right) \times \exp\left(-\frac{\mathrm{i}}{\hbar}\sum_{p=1}^{N} \frac{W_p(X, Y, Z)}{p}\epsilon^p\right)$$

expand the result in power series of ϵ and collect all coefficients multiplying ϵ^ℓ. The result is

$$\begin{aligned} &\mathrm{i}\hbar\mathcal{V}_\ell(n, \hbar; X, Y, Z) = \mathrm{Im}\,\nabla_{X,Y,Z} \wedge \sum_{s=1}^{N} \frac{1}{s} \sum_{k_1+\cdots+k_s=\ell-s} \hbar^{-s} G_{k_1,\ldots,k_s}(X, Y, Z) \\ &G_{k_1,\ldots,k_s}(X, Y, Z) = \langle f_n, [W_{k_1}, [W_{k_2}, [\ldots, [W_{k_s}, \nabla_{X,Y,Z} W_s] \ldots] f_n\rangle. \end{aligned} \tag{3.22a}$$

Inserting s times (3.18a,b) to compute the sth commutator on account of the quantum normal form (3.19b) for W_j, and hence for $\nabla_{X,Y,Z}W_j$, we eliminate the power $\hbar^{-s}$, and each addend $G_{k_1,\ldots,k_s}(X,Y,Z)$ takes the form

$$G_{k_1,\ldots,k_s}(X,Y,Z) = \hbar^q \langle f_n, \widehat{F}_{X,Y,Z}(\zeta,\eta) f_n\rangle \tag{3.22b}$$

for some $q \geqslant 0$ and some polynomial symbol $F_{X,Y,Z}(\zeta,\eta)$ depending holomorphically on (X,Y,Z). Now, by (3.16f)

$$G_{k_1,\ldots,k_s}(X,Y,Z) = \hbar^q \langle e_n, \mathcal{U}^{-1}\widehat{F}_{X,Y,Z}(\zeta,\eta)\mathcal{U} e_n\rangle. \tag{3.22c}$$

Now, once more by [16, section 3f] (see also the discussion in [17, section 5.1.4]), since the Hamiltonians $H_0(\zeta,\eta)$ and $H_0(\zeta',\eta')$ are related through the linear canonical change of variables (2.7a), and the basis vectors $\{e_n(z) : n \in \mathbb{N}\cup\{0\}^l\}$ are the eigenvectors of $S_0' := \hbar\langle\omega_z, \nabla_z\rangle$, the canonical quantization of $H_0(\zeta',\eta')$ in the coordinates (ζ',η'), the operator $\mathcal{U}^{-1}\widehat{F}_{X,Y,Z}(\zeta,\eta)\mathcal{U}$ is the quantization of the symbol $F(X,Y,Z)$ composed with (2.7a), which depends holomorphically on (X,Y,Z). Namely

$$\mathcal{U}^{-1}\widehat{F}_{X,Y,Z}(\zeta,\eta)\mathcal{U} = \widehat{F}_{X,Y,Z}(\zeta'(\zeta,\eta),\eta'(\zeta,\eta); X,Y,Z) \tag{3.22d}$$

hence

$$G_{k_1,\ldots,k_s}(X,Y,Z) = \hbar^q \langle e_n, \widehat{F}_{X,Y,Z}(\zeta'(\zeta,\eta),\eta'(\zeta,\eta); X,Y,Z) e_n\rangle. \tag{3.22e}$$

Now the scalar product in the RHS of this last expression is a polynomial in $\hbar$, with holomorphic dependence of the coefficients on X, Y, Z, by a direct application of lemma 3 of [11]. Therefore each addend of the equation (3.16d) for $\hbar\gamma_\ell(n;C)$ is explicitly displayed as a polynomial in $\hbar$ with holomorphic dependence of the coefficients on (X,Y,Z) and this proves the assertion. □

To conclude the proof of proposition 1 we have to identify the coefficients $\gamma_\ell^j(n;C)$ with the coefficients $\Gamma_\ell^j(n\hbar;C)$ of (1.9) and to prove the relation (1.10).

Proof of proposition 1, part (3). To prove the validity of the expansion (1.9) we have to show that the coefficients $\gamma_\ell^j(n;C)$ of (3.21) depend on n and $\hbar$ only through the combination $n\hbar$; in turn, by (3.22) we can limit ourselves to verifying this property on the matrix elements $G_{k_1,\ldots,k_s}(X,Y,Z)$. Now, once more by [11, lemma 3, equation (3.13)] we can write

$$\begin{aligned} G_{k_1,\ldots,k_s}(X,Y,Z) &= \langle e_n, \widehat{F}_{X,Y,Z}(\zeta'(\zeta,\eta),\eta'(\zeta,\eta); X,Y,Z) e_n\rangle \\ &= (F_{X,Y,Z})_0(I)|_{I=n\hbar} + \sum_{j=1}^{\deg F} \hbar^j \sum_{|\alpha|=j} D^{(\alpha)}(F_{X,Y,Z})_0(I)|_{I=n\hbar} \end{aligned} \tag{3.23}$$

where $(F_{X,Y,Z})_0(I)$ denotes the zero Fourier coefficient of $F_{X,Y,Z}$ re-expressed in the variables $I_k = \zeta_k\eta_k, \theta_k = \arg\zeta_k$, and $D^{(\alpha)}$ is the differential operator introduced in [11, equations (2.32)–(2.34)] whose definition need not to be recalled here. (3.23) displays the desired dependence, and this proves (1.9).

To conclude the proof of part (3), let us first identify the coefficient of order zero in $\hbar$ in (3.21). By (3.18a, b) the term of order zero in $\hbar$ in (3.22a) has the form

$$i\hbar\mathcal{V}_\ell(n,0;X,Y,Z)$$
$$= \left\langle f_n, \sum_{s=1}^{N} \sum_{k_1+\cdots+k_s=\ell-s} \{w_{k_1}, \{w_{k_2}, \{\ldots, \{w_{k_s}, \nabla_{X,Y,Z} w_s\} \ldots\}^{\widehat{}} f_n \right\rangle \tag{3.24}$$

and, once more by equation (3.13) of [11], it is a polynomial in $\hbar$ whose term of order zero is

$$i\hbar\mathcal{V}_\ell^0(n;X,Y,Z) = \operatorname{Im} \nabla_{X,Y,Z}$$
$$\wedge \left(\sum_{s=1}^{N} \sum_{k_1+\cdots+k_s=\ell-s} \{w_{k_1}, \{w_{k_2}, \{\ldots, \{w_{k_s}, \nabla_{X,Y,Z} w_s\} \ldots\} \right)_0 (I)_{I=n\hbar}. \tag{3.25}$$

On the other hand

$$\sum_{s=1}^{N} \sum_{k_1+\cdots+k_s=\ell-s} \{w_{k_1}, \{w_{k_2}, \{\ldots, \{w_{k_s}, \nabla_{X,Y,Z} w_s\} \ldots\}$$
$$= \frac{1}{\ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}\epsilon^\ell}|_{\epsilon=0} \mathrm{e}^{\mathcal{L}_{w_N(\epsilon)}} \nabla_{X,Y,Z} w_N(\epsilon) \tag{3.26}$$

where, as above

$$w_N(\epsilon) = \sum_{k=1}^{N} w_k \epsilon^k.$$

Now, by construction

$$\mathrm{e}^{\mathcal{L}_{w_N(\epsilon)}} \nabla_{X,Y,Z} w_N(\epsilon) = \nabla_{X,Y,Z} w_N(I(\epsilon), \theta(\epsilon), \epsilon) + \mathrm{O}(\epsilon^{N+1}).$$

Hence the RHS of (3.26) is just the ℓth term of the canonical perturbation expansion for $\nabla_{X,Y,Z} w(\epsilon)$ computed on the canonical coordinates (I, θ). We now remark that, on account of the quantization condition, $I_k = n_k\hbar$, we have $\hbar\partial_{I_k} = \partial_{n_k}$, while, as in (2.22), (2.23), we obtain

$$\frac{\partial}{\partial I_k}\frac{1}{\ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}\epsilon^\ell}|_{\epsilon=0} \nabla_{X,Y,Z} w(\epsilon) = \nabla_{X,Y,Z} q_k^\ell(I,\theta;X,Y,Z).$$

The average over θ is performed by inserting this in the RHS of (3.25). Performing the further external differentiation with respect to (X, Y, Z) in (3.25) we can apply without any change the original Berry argument [3, equations (15)–(18), (28)–(30)] to conclude that

$$i\hbar\partial_{I_k}\mathcal{V}_\ell^0(n;X,Y,Z) \equiv \partial_{n_k}\mathcal{V}_\ell^0(n;X,Y,Z) \equiv \partial_{n_k}\Gamma_\ell^0(n,C) = a_\ell^k(I,C).$$

The argument so far exposed can be repeated without changes for all terms multiplying the increasing powers of $\hbar$, and this remark concludes the proof of proposition 1. □

4. Discussion

Berry's geometric phase connects the physical consequences of adiabatic changes to the underlying topology in parameter space, and its importance in several areas of physics (beyond its natural occurrence in the quantum Hall effect and in the Bohm–Aharonov effect) is well known (see e.g. [18, 19]). The Berry phase has a natural classical counterpart, the Hannay angle, and therefore determining if and how it reduces to Hannay's angle in the classical limit is an interesting problem of semiclassical analysis, first taken on by Berry himself [3]: he established the connection formula (1.1) through Bohr–Sommerfeld quantization in the case of the generalized harmonic oscillator (which represents the simplest example of a one-dimensional system admitting a non-trivial topology in the parameter space). The Bohr–Sommerfeld quantization, however, is applicable only to classically integrable systems, while Hannay's angle can also be defined for non-integrable ones [20, 21]. Hence the problem of extending (1.1) to multi-dimensional systems, which are generically non-integrable, and even of determining the very existence of the classical limit of Berry's phase is far from trivial. Existing results are therefore limited to the integrable case and hence very partial.

It is however well known that, under certain circumstances, non-integrable systems which can be written as a perturbation of integrable ones, $H(\epsilon) = H_0 + \epsilon V$, admit normal forms, i.e. classical perturbation expansion to all orders in ϵ. Replacing $H(\epsilon)$ by its normal form amounts to replacing $H(\epsilon)$ through a (canonically equivalent) integrable Hamiltonian up to an error of order ϵ^∞, i.e. having zero asymptotic expansion in ϵ. For perturbations of harmonic oscillators, this normal form is most conveniently constructed in the complex canonical coordinates $(\zeta, \eta) := (1/\sqrt{2})(Q + \mathrm{i}P, Q - \mathrm{i}P)$. This last class of systems also admits a quantum normal form. Namely, the quantum perturbation theory, in which the coefficient of ϵ^k is given by the exact quantization of the corresponding classical one. By this we mean that we can recursively construct classical functions $f_j^k(\zeta, \eta) : f_0^k(\zeta, \eta) \equiv f^k(\zeta, \eta)$ such that:

(a) The function (called the full classical symbol)

$$F^k(\zeta, \eta) = f^k(\zeta, \eta) + \sum_{j=1}^{\infty} f_j^k(\zeta, \eta)\hbar^j$$

yields the quantum coefficient $\widehat{F_k}$ upon Weyl quantization in the Bargmann representation.

(b) The function $f^k(\zeta, \eta)$ is the corresponding term of the classical expansion.

Replacing the quantum operator with its normal form amounts to replacing it with a unitarily equivalent one which is diagonal on the unperturbed basis up to an error which has zero asymptotic expansion in ϵ *uniformly with respect to* $\hbar$. Therefore if we replace the quantum and classical Hamiltonians by their normal forms, respectively, we are reduced to comparing a diagonal quantum operator with an integrable classical Hamiltonian up to an error having vanishing asymptotic expansion independently of $\hbar$.

This method, if applicable, yields by construction the correct semiclassical expansion. What has been done in this paper is to examine the relation between Berry's phase and Hannay's angle within this technique. The addition to existing theory obtained here seems to be twofold: on one side, the construction of the full semiclassical expansion of Berry's phase to all orders in perturbation theory for a large class of non-integrable systems, and on the other side the adaptation of the construction of the quantum normal form to account for the dependence on the parameters needed to compute Berry's phase: in particular this requires a verification of the general fact that full symbols in perturbation theory (not only the principal ones) related by linear canonical transformations generate unitarily equivalent operators upon quantization.

Since the normal forms are divergent (uniformly with respect to $\hbar$) the limitations of the present result are the usual ones encountered in substituting a formal power expansion for the exact solution. Neglecting a remainder term even with zero asymptotic expansion in ϵ amounts in classical mechanics to saying that, for ϵ small, all tori will be conserved (while we know from KAM theory that some of them disappear no matter how small ϵ is taken); in quantum mechanics this amounts to saying that for equally small ϵ all levels can be somehow approximated by a Bohr–Sommerfeld quantization formula plus corrections in ascending powers of $\hbar$, while we know that some of them do not admit such approximation no matter how small ϵ is. On the other hand the elimination of this $O(\epsilon^{\infty})$ error would require a formulation of KAM theory valid uniformly with respect to $\hbar$, and to our knowledge this problem seems to be completely open.

References

[1] Berry M V 1984 Quantal phase factors accompanying adiabatic changes *Proc. R. Soc.* A **392** 45–57

[2] Hannay J H 1985 Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian *J. Phys. A: Math. Gen.* **18** 221–30

[3] Berry M V 1985 Classical adiabatic angles and quantal adiabatic phase *J. Phys. A: Math. Gen.* **18** 15–27

[4] Asch J 1990 On the classical limit of Berry's phase for integrable systems *Commun. Math. Phys.* **127** 637–51

[5] Gérard C and Robert D 1990 La limite semiclassique de la phase de Berry *C. R. Acad. Sci. Paris* **310** 677–82

[6] Gérard C and Robert D 1990 On the semiclassical asymptotics of Berry's phase *Séminaire EDP* Université de Nantes

[7] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics* vol IV (New York: Academic)

[8] Simon B 1970 Coupling constant analyticity for the anharmonic oscillator *Ann. Phys., NY* **57** 79–130

[9] Avron Y, Seiler R and Yaffe L G 1987 Adiabatic theorems and application to the quantum Hall effect *Commun. Math. Phys.* **82** 33–49

[10] Graffi S and Paul T 1987 The Schrödinger equation and canonical perturbation theory *Commun. Math. Phys.* **108** 71–87

[11] Degli Esposti M, Graffi S and Herczynski J 1991 Quantization of the classical Lie algorithm in the Bargmann representation *Ann. Phys., NY* **209** 364–93

[12] Bellissard J and Vittot M 1990 Heisenberg's picture and non-commutative geometry of the semiclassical limit in quantum mechanics *Ann. Inst. Henri Poincaré* B **52** 15–27

[13] Eckhardt B 1986 Lie transform Birkhoff–Gustafson normal form in classical and quantum mechanics *J. Phys. A: Math. Gen.* **18** 2961–72

[14] Cary J R 1981 Lie transform perturbation theory for Hamiltonian systems *Phys. Rep.* **79** 129–59

[15] Giorgilli A and Galgani L 1985 Rigorous estimates for the series expansions of Hamiltonian perturbation theory *Celestial Mech.* **37** 95–112

[16] Bargmann V 1961 On a Hilbert space of analytic functions and an associated integral transform *Commun. Pure Appl. Math.* **14** 187–214

[17] Berezin F A and Shubin M A 1991 *The Schrödinger Equation* (Dordrecht: Kluwer)

[18] Shapere A and Wilczek F 1989 *Geometric Phases in Physics* (Singapore: World Scientific)

[19] Jackiw R 1988 Berry's phase—topological ideas from atomic, molecular and optical physics *Commun. Atom. Mol. Phys.* **21** 71–82

[20] Montgomery R 1988 The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case *Commun. Math. Phys.* **120** 269–94

[21] Golin S, Knauf A and Marmi S 1989 The Hannay angle: geometry, adiabaticity and an example *Commun. Math. Phys.* **123** 95–122